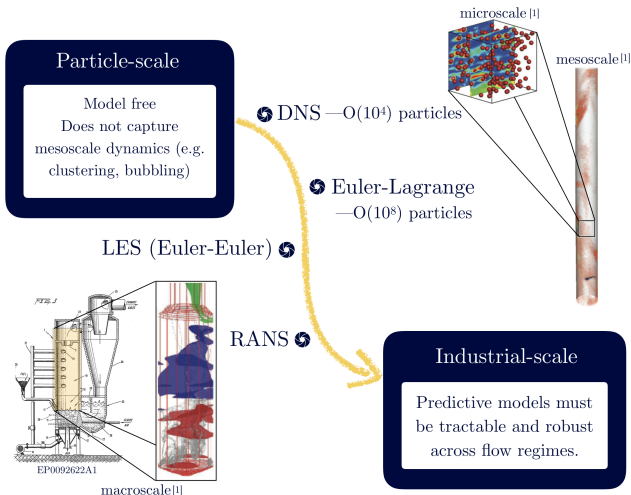


Sparse identification of multiphase RANS closures

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Challenges in multiphase turbulence modeling



[1] Tenneti and Subramaniam (2014)

Objectives and outline

Objective: Develop accurate and predictive multiphase turbulence models that capture relevant physics across scales.

Agenda:

- Canonical flow to isolate 2-way coupling in turbulence
 - ▶ Fully-developed cluster induced turbulence (CIT)
 - ▶ Numerical framework for generating data
 - ▶ Shortcomings of existing closure models
- The role of machine learning in turbulence modeling
- Sparse identification
 - ▶ Application to single-phase turbulence
 - ▶ Application to fully-developed CIT
- Looking ahead

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Fully-developed, cluster-induced turbulence ¹ ²

- Requires a mean velocity difference between phases (e.g. body forces (gravity), inlet conditions)
- Mass loading $\varphi = \frac{\rho_p}{\rho_f} \frac{\langle \alpha_p \rangle}{\langle \alpha_f \rangle} \gtrsim 1$
- Spontaneous cluster formation
 - ▶ Characteristic timescale $\tau_p = \frac{\rho_p}{\rho_f} \frac{d_p^2}{18\nu_f}$
 - ▶ Terminal velocity $\mathcal{V} = \tau_p g$
 - ▶ Particle Reynolds number $\text{Re}_p = \frac{\mathcal{V} d_p}{\nu_f}$
 - ▶ Cluster size $\mathcal{L} = \tau_p^2 g \gg d_p$
- Fluctuations in α_p can generate and sustain fluid-phase turbulence (fully-developed CIT)
- In the absence of mean shear, turbulent kinetic energy is produced by two-way coupling through drag (Drag Production).

We consider the most
simplified two-way
coupled flow.

[1] J. Capecelatro *et al.* (2014), *J. Fluid Mech.*, [2] Agrawal, Sundaresan, *et al.* (2001), *J. Fluid Mech.*

Eulerian-Lagrangian solution of mesoscale equations

- NGA³

- ▶ Finite volume DNS/LES code
- ▶ Conservation of mass, momentum and kinetic energy

Interphase momentum exchange

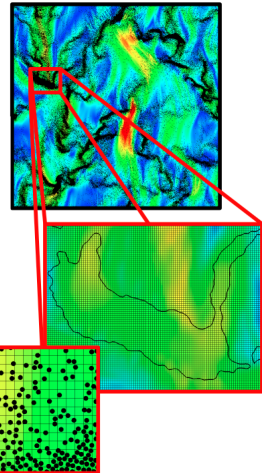
$$\frac{\partial \alpha_f \mathbf{u}_f}{\partial t} + \nabla \cdot \alpha_f \mathbf{u}_f \otimes \mathbf{u}_f = -\frac{1}{\rho_f} \nabla p_f + \frac{1}{\rho_f} \nabla \cdot \boldsymbol{\sigma}_f - \alpha_f \varphi \mathcal{A} + \alpha_f \mathbf{g}$$

- Lagrangian particle tracking⁴

- ▶ Newton's 2nd law for particle position and velocity
- ▶ 2nd-order Runge-Kutta for particle ODEs
- ▶ Soft-sphere collision model

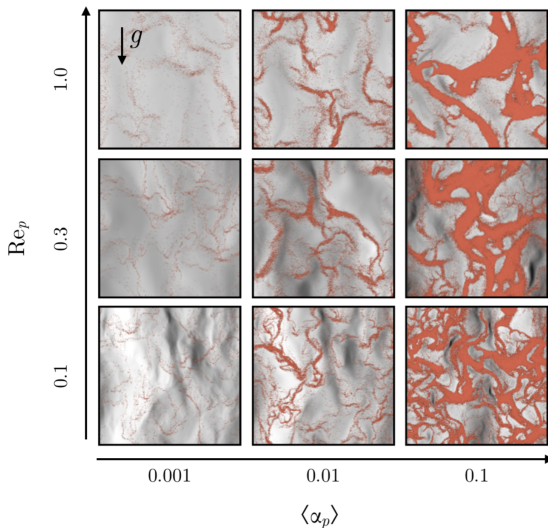
- Interphase exchange

- ▶ Particle volume fraction and momentum transferred to fluid
- ▶ Fully conservative and consistent filtering approach⁴
- ▶ Transferred data converges under mesh refinement



[3] Desjardins *et al.* (2008) [4] Capecelatro *et al.* (2013)

Simulation results



- Mesh: 512×512 ($896 d_p \times 896 d_p$)
- $N_p = (12,359 \quad 57,362 \quad 266,278)$
- Enforce mass flow rate = 0

RANS equations for fully-developed CIT

Phase-averaging of the mesoscale equations allow us to arrive at the RANS equations for fully-developed CIT.^{5 6}

$$\begin{aligned}
 \frac{1}{2} \frac{\partial \langle u_f'''^2 \rangle_f}{\partial t} = & \frac{1}{\rho_f} \left(\underbrace{\left\langle p_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right) \\
 & + \underbrace{\frac{\varphi}{\tau_p} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f'''^2 \rangle_p \right)}_{\text{DE}} \\
 & + \underbrace{\frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p}_{\text{DP}} + \frac{\varphi}{\rho_p} \left(\underbrace{\left\langle u_f''' \frac{\partial p'_f}{\partial x} \right\rangle_p}_{\text{PE}} - \underbrace{\left\langle u_f''' \frac{\partial \sigma'_{f,xi}}{\partial x_i} \right\rangle_p}_{\text{VE}} \right)
 \end{aligned}$$

RANS equations for fully-developed CIT

$$\frac{1}{2} \frac{\partial \langle u_f'''^2 \rangle_f}{\partial t} = \frac{1}{\rho_f} \left(\underbrace{\left\langle \rho_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right)$$

$$u_f''' = u_f - \langle u_f \rangle_f$$

fluctuations in fluid velocity

fluid-phase averaged
fluid velocity $\rightarrow \langle u_f \rangle_f = \frac{\langle \alpha_f u_f \rangle}{\langle \alpha_f \rangle}$

$\langle u_f'''^2 \rangle_f \leftarrow$ fluid-phase Reynolds-Stress

$$\underbrace{\left\langle u_f''' \frac{\partial \sigma'_{f,xi}}{\partial x_i} \right\rangle_p}_{\text{VE}}$$

RANS equations for fully-developed CIT

$$\frac{1}{2} \frac{\partial \langle u_f'^2 \rangle_f}{\partial t} = \frac{1}{\rho_f} \left(\underbrace{\left\langle p_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right) + \frac{\varphi}{2} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f''^2 \rangle_p \right)$$

Fluid-phase Pressure Strain & Viscous Dissipation

- Present in single-phase flows.
- PS produces TKE in the horizontal direction due to weak reorientation of vertical to horizontal Reynolds stress.
- VD dissipates TKE to heat
- $\sigma_{f,xi}$ is the viscous stress tensor.

RANS equations for fully-developed CIT

Drag Production

- Source of fluid-phase TKE
- Only present in coupled flows and in the gravity-aligned direction.
- In the absence of clustering, particles are uncorrelated and $\langle u_f''' \rangle_p = 0$.

particle-phase average $\langle (\cdot) \rangle_p = \frac{\langle \alpha_p (\cdot) \rangle}{\langle \alpha_p \rangle}$

$\langle u_f''' \rangle_p$ fluid velocity fluctuation
seen by the particles

$$\underbrace{+ \frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p}_{\text{DP}} + \underbrace{\frac{\varphi}{\tau_p} \left(\left\langle u_f \frac{\partial p'_f}{\partial x} \right\rangle_p \right)}_{\text{PE}} - \underbrace{\left\langle u_f''' \frac{\partial \sigma'_{f,xi}}{\partial x_i} \right\rangle_p}_{\text{VE}}$$

RANS equations for fully-developed CIT

$$\frac{1}{2} \frac{\partial \langle u_f'''^2 \rangle_f}{\partial t} = \frac{1}{\rho_f} \left(\underbrace{\left\langle p_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right)$$

$$+ \underbrace{\frac{\varphi}{\tau_p} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f'''^2 \rangle_p \right)}_{\text{DE}}$$

Fluid-phase Drag Exchange

$\langle u_f''' u_p'' \rangle_p$ fluid-particle covariance
seen by the particles

$\frac{\varphi}{\tau_p}$ mass loading

characteristic time scale

DE partitions TKE
between fluid and
particle phases.

RANS equations for fully-developed CIT

$$\begin{aligned}
 \frac{1}{2} \frac{\partial \langle u_f'^2 \rangle_f}{\partial t} &= \frac{1}{\rho_f} \underbrace{\left(\frac{\partial \langle p_f' \rangle_f}{\partial t} + \frac{\partial \langle \sigma_f' \rangle_f}{\partial t} \right)}_{\text{DE}} \\
 &+ \underbrace{\frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p}_{\text{DP}} + \underbrace{\frac{\varphi}{\rho_p} \left(\underbrace{\left\langle u_f''' \frac{\partial p_f'}{\partial x} \right\rangle_p}_{\text{PE}} - \underbrace{\left\langle u_f''' \frac{\partial \sigma_{f,xi}'}{\partial x_i} \right\rangle_p}_{\text{VE}} \right)}_{\text{PE and VE}}
 \end{aligned}$$

Pressure and Viscous Exchange

- Since the particle-to-fluid density ratio is large, these terms are often neglected.
- p_f' is the fluid pressure fluctuation
- σ_f' is the viscous stress fluctuation

Existing closure models

Unclosed terms are typically modeled by extension from single-phase flow⁴

Pressure Strain & Viscous Dissipation:

$$\frac{1}{\rho_f} \left(\left\langle \rho_f \frac{\partial u_f'''}{\partial x} \right\rangle - \left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle \right) = -C_{Rf} \frac{\varepsilon_f}{k_f} \left(\langle u_f'''^2 \rangle_f - k_f \right) - \varepsilon_f$$

Drag Exchange:

$$\frac{\varphi}{\tau_p} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f'''^2 \rangle_p \right) = \frac{\varphi}{\tau_p} \left(\beta_{fp,1} \langle u_f'''^2 \rangle_f^{1/2} \langle u_p''^2 \rangle_p^{1/2} - \beta_{f,1} \langle u_f'''^2 \rangle_f \right)$$

Drag Production:

$$\frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p = \frac{\varphi}{\tau_p} C_g \langle u_p \rangle_p^2$$

$$\text{where } C_g = \frac{\langle u_f''' \rangle_p}{\langle u_p \rangle_p}$$

Pressure and Viscous Exchange Terms are assumed to be negligible.

[4] Fox, R. (2014); Capecelatro *et al.* (2015)

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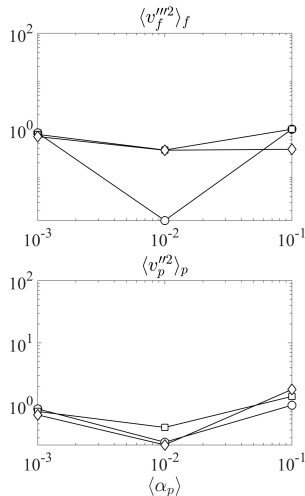
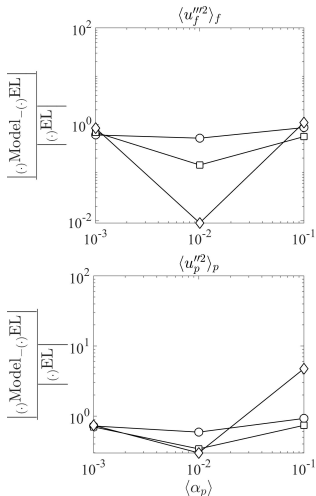
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Existing closures fail to be predictive



$Re_p = 0.1 : \bigcirc, Re_p = 0.3 : \blacklozenge, Re_p = 1.0 : \square$

Closures adapted from single-phase break down for even the most simplified two-way coupled flows

Can machine learning provide a
tractable closure?

Machine learning in CFD

Neural Networks⁵ and **Sparse Regression**⁶ are key players in data-driven turbulence modeling.

Neural Networks

Benefits

- Complete generality
- Requires no prior knowledge of underlying physics

Drawbacks

- Difficult to express in compact form

Sparse Regression

Benefits

- Compact, algebraic form
- Model selection *and* regression
- Easily incorporated into CFD solvers

Drawbacks

- Relies on modeler to select appropriate test functions

[5] Duraysami *et al* (2018) [6] Brunton *et al* (2016)

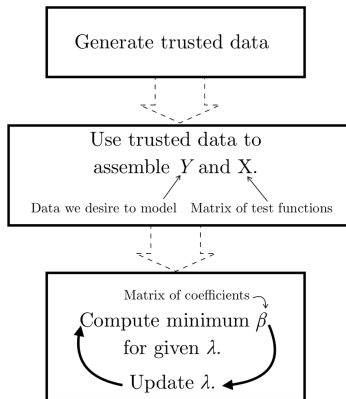
Sparse identification

We define the sparse model as

$$\mathbf{y} = \mathbf{X}\hat{\beta}.$$

Determining the optimal coefficient matrix is cast as a sparse minimization problem

$$\hat{\beta} = \min_{\beta} \underbrace{\|\mathbf{y} - \mathbf{X}\beta\|_2^2}_{\text{regression}} + \underbrace{\lambda\|\beta\|_1}_{\text{sparsity}}$$



Demonstrated success for classical, dynamic systems (e.g. the Lorenz attractor).⁶

Can this technique be applied successfully to realistic flows?

[6] Brunton *et al* (2016)

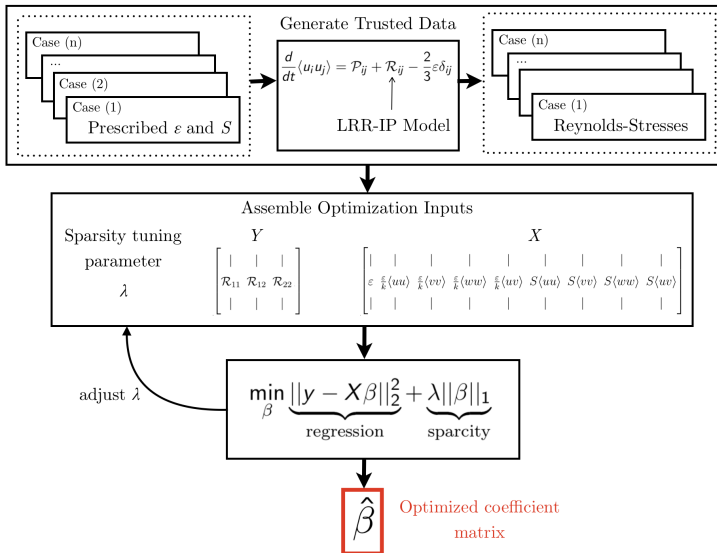
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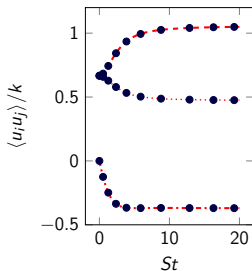


Sparse Regression method applied to synthetic data

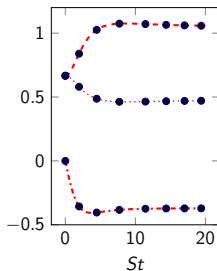


Sparse identification correctly returns the LRR-IP model

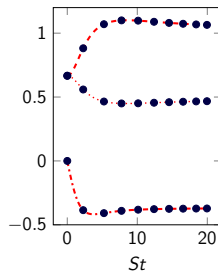
Equation	ε	$\frac{\varepsilon}{k} \langle uu \rangle$	$\frac{\varepsilon}{k} \langle uv \rangle$	$\frac{\varepsilon}{k} \langle vv \rangle$	$\langle uu \rangle \frac{\partial \langle u \rangle}{\partial y}$	$\langle uv \rangle \frac{\partial \langle u \rangle}{\partial y}$	$\langle uv \rangle \frac{\partial \langle u \rangle}{\partial y}$
$\mathcal{R}_{1,1}$	$\frac{2}{3} C_R$	$-C_R$	0	0	0	0	$\frac{4}{3} C_2$
$\mathcal{R}_{1,1}^{\text{SIMR}}$	1.2056	-1.8054	0	0	0	0	0.79999
$\mathcal{R}_{1,2}$	0	0	$-C_R$	0	0	C_2	0
$\mathcal{R}_{1,2}^{\text{SIMR}}$	0	0	-1.79956	0	0	0.600055	0
$\mathcal{R}_{2,2}$	$\frac{2}{3} C_R$	0	0	$-C_R$	0	0	$-\frac{2}{3} C_2$
$\mathcal{R}_{2,2}^{\text{SIMR}}$	1.197513	0	0	-1.79956	0	0	-0.40038


 $S = 3$

$$\|E^{\text{SR}}\|_{\infty} = 1.2 \times 10^{-4}$$

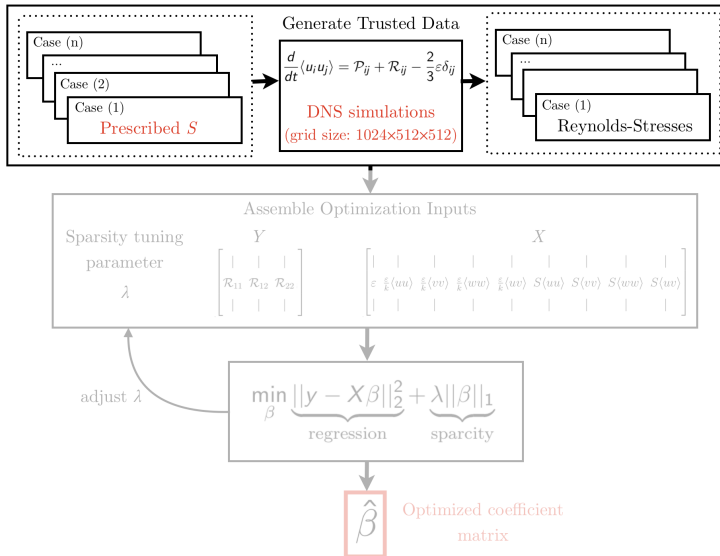

 $S = 15$

$$\|E^{\text{SR}}\|_{\infty} = 7.8 \times 10^{-6}$$


 $S = 27$

$$\|E^{\text{SR}}\|_{\infty} = 4.1 \times 10^{-4}$$

DNS-generated shear turbulence



DNS-generated homogeneous shear turbulence

Discovered sparse identification model

Equation	ε	$\frac{\varepsilon}{k} \langle uu \rangle$	$\frac{\varepsilon}{k} \langle vv \rangle$	$\frac{\varepsilon}{k} \langle ww \rangle$	$\frac{\varepsilon}{k} \langle uv \rangle$	$S \langle uu \rangle$	$S \langle vv \rangle$	$S \langle ww \rangle$	$S \langle uv \rangle$
$\mathcal{R}_{1,1}^{\text{LLR-IP}}$	$\frac{2}{3} C_R$	$-C_R$	0	0	0	0	0	0	$-\frac{4}{3} C_2$
$\mathcal{R}_{1,1}^{\text{SIMR}}$	43.32	-20.07	0	-48.98	-20.36	0	-1.95	2.03	3.04
$\mathcal{R}_{1,2}^{\text{LLR-IP}}$	0	0	$-C_R$	0	0	$-C_2$	0		
$\mathcal{R}_{1,2}^{\text{SIMR}}$	10.38	-6.37	0	-2.70	8.74	0	0.65	0	-0.45
$\mathcal{R}_{2,2}^{\text{LLR-IP}}$	$\frac{2}{3} C_R$	0	0	$-C_R$	0	0	$\frac{2}{3} C_2$		
$\mathcal{R}_{2,2}^{\text{SIMR}}$	-5.09	3.38	0	-4.99	-17.51	0	0.42	0	0.41
$\mathcal{R}_{3,3}^{\text{LLR-IP}}$	$\frac{2}{3} C_R$	0	0	$-C_R$	0	0	$\frac{2}{3} C_2$		
$\mathcal{R}_{3,3}^{\text{SIMR}}$	-11.17	8.12	0	-4.14	-21.29	0	0.81	0	0.99

LLR-IP model⁷

$$\mathcal{R}_{ij} = -C_R \frac{\varepsilon}{k} \left(\langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij} \right) - C_2 \left(\mathcal{P}_{ij} - \frac{2}{3} \mathcal{P} \delta_{ij} \right)$$

with typical values $C_R = 1.8$ and $C_2 = 3/5$.

Model identified by sparse identification

$$\mathcal{R}_{1,1} = 28 \frac{\varepsilon}{k} (\langle uu \rangle + 5/3 \langle vv \rangle - 0.7 \langle uv \rangle - 1.9k) - 2S \langle vv \rangle + 2S \langle ww \rangle + 3S \langle uv \rangle$$

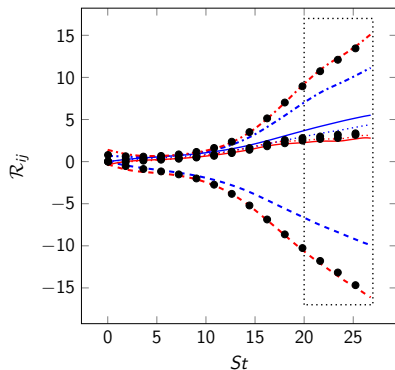
$$\mathcal{R}_{1,2} = 2.7 \frac{\varepsilon}{k} (-1.4 \langle uu \rangle + \langle vv \rangle + 3.2 \langle uv \rangle + 1.9k) + 0.6S \langle vv \rangle - 0.4S \langle uv \rangle$$

$$\mathcal{R}_{2,2} = 5 \frac{\varepsilon}{k} (1.67 \langle uu \rangle + \langle vv \rangle - 3.5 \langle uv \rangle - 3k) - 0.4S \langle vv \rangle + 0.4S \langle uv \rangle$$

$$\mathcal{R}_{3,3} = 4 \frac{\varepsilon}{k} (3 \langle uu \rangle + \langle vv \rangle - 5 \langle uv \rangle - 4.7k) - 0.8S \langle vv \rangle + S \langle uv \rangle$$

DNS-generated homogeneous shear turbulence

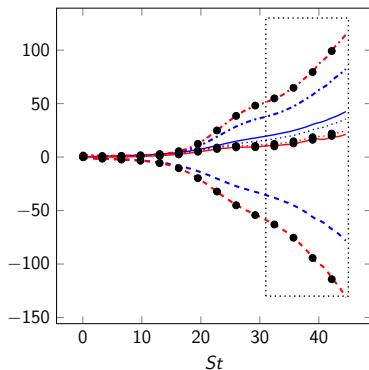
Sparse identification model out performs LRR-IP model



$S = 15$

$$\|E^{\text{LRR-IP}}\|_{\infty} = 0.551$$

$$\|E^{\text{SR}}\|_{\infty} = 0.106$$



$S = 27$

$$\|E^{\text{LRR-IP}}\|_{\infty} = 0.523$$

$$\|E^{\text{SR}}\|_{\infty} = 0.107$$

Sparse identification

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Sparse identification of fully-developed CIT

We group the unclosed terms and conduct sparse identification for the four terms using highly-resolved data generated using NGA.

$$\begin{aligned}
 \frac{1}{2} \frac{\partial \langle u_f'''^2 \rangle_f}{\partial t} = & \quad \textcircled{1} \quad \frac{1}{\rho_f} \left(\underbrace{\left\langle p_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right) \\
 & \quad \textcircled{2} \quad + \underbrace{\frac{\varphi}{\tau_p} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f'''^2 \rangle_p \right)}_{\text{DE}} \\
 & \quad \textcircled{3} \quad + \underbrace{\frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p}_{\text{DP}} + \textcircled{4} \quad \frac{\varphi}{\rho_p} \left(\underbrace{\left\langle u_f''' \frac{\partial p'_f}{\partial x} \right\rangle_p}_{\text{PE}} - \underbrace{\left\langle u_f''' \frac{\partial \sigma'_{f,xi}}{\partial x_i} \right\rangle_p}_{\text{VE}} \right)
 \end{aligned}$$

Sparse identification of fully-developed CIT

$$\frac{1}{\rho_f} \left(\underbrace{\left\langle p_f \frac{\partial u_f'''}{\partial x} \right\rangle}_{\text{PS}} - \underbrace{\left\langle \sigma_{f,xi} \frac{\partial u_f'''}{\partial x_i} \right\rangle}_{\text{VD}} \right) = C_{Rf} \frac{\varepsilon_f}{k_f} \left(\langle u_f'''^2 \rangle_f - k_f \right) - \varepsilon_f$$

Exact expression

Existing Closure

Closure Discovered by Sparse Regression

$$= -1.1 \frac{\varepsilon_f}{k_f} \langle u_f'''^2 \rangle + \frac{\varphi}{\tau_p} \left(0.004 \langle v_f'''^2 \rangle + (0.02 Re_p + 0.14 \langle \alpha_p \rangle - 0.02) \langle u_f'''^2 \rangle \right) \\ + \frac{\varphi}{\tau_p} \left((0.06 Re_p + 0.05) \langle u_p''^2 \rangle_p - 0.13 \langle v_p''^2 \rangle_p \right) - 0.13 Re_p \frac{\varphi}{\tau_p} \langle u_f'''^2 \rangle^{\frac{1}{2}} \langle u_p''^2 \rangle_p^{\frac{1}{2}}$$

Existing Model Error

		0.001	0.01	0.1
Re_p	0.1	0.66	0.83	0.89
	0.3	0.53	0.92	0.97
	1.0	0.77	0.92	0.98

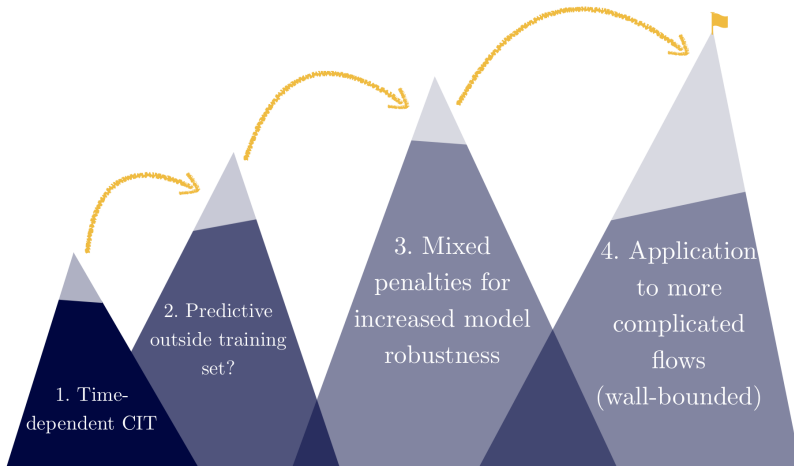
Sparse identification Error

		0.001	0.01	0.1
Re_p	0.1	0.7×10^{-14}	0.3×10^{-14}	0.1×10^{-14}
	0.3	0.7×10^{-14}	0.1×10^{-14}	0.1×10^{-14}
	1.0	0.3×10^{-14}	0.1×10^{-14}	0.1×10^{-14}



Sparse Identification uncovers complex dependencies on flow conditions that improve model accuracy.

Looking ahead



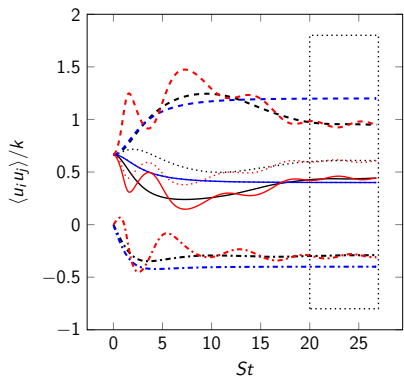
Questions?



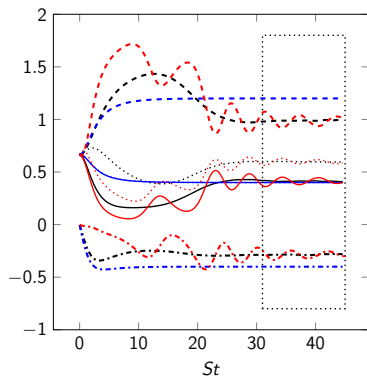
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DNS-generated homogeneous shear turbulence

Comparison of models in the context of the Reynolds-stress equations



$S = 15$



$S = 27$

Sparse identification of fully-developed CIT

$$\underbrace{\frac{\varphi}{\tau_p} \left(\langle u_f''' u_p'' \rangle_p - \langle u_f'''^2 \rangle_p \right)}_{\text{DE}} = \frac{\varphi}{\tau_p} \left(\beta_{fp,1} \langle u_f'''^2 \rangle_f^{1/2} \langle u_p''^2 \rangle_p^{1/2} - \beta_{f,1} \langle u_f'''^2 \rangle_f \right)$$

Exact expression

Existing Closure

Closure Discovered by Sparse Regression

$$= 9.1 \frac{\varphi_f}{k_f} \langle u_f'''^2 \rangle - \frac{\varphi}{\tau_p} \left((0.5 + 1.9 \langle \alpha_p \rangle + 0.05 Re_p) \langle u_f'''^2 \rangle + 0.1 \langle v_f'''^2 \rangle \right) \\ + \frac{\varphi}{\tau_p} \left((0.6 - 0.05 Re_p) \langle u_p''^2 \rangle_p - 0.5 \langle v_p''^2 \rangle_p \right) - 0.4 Re_p \frac{\varphi}{\tau_p} \langle u_f'''^2 \rangle^{\frac{1}{2}} \langle u_p''^2 \rangle_p^{\frac{1}{2}}$$

Existing Model Error

		0.001	$\langle \alpha_p \rangle$ 0.01	0.1
Re_p	0.1	0.24	0.05	1.23
	0.3	0.72	0.21	0.32
	1.0	0.35	0.25	0.24

Sparse Identification Error

		0.001	$\langle \alpha_p \rangle$ 0.01	0.1
Re_p	0.1	0.6×10^{-14}	0.2×10^{-14}	0.2×10^{-14}
	0.3	4.9×10^{-14}	0.2×10^{-14}	0.4×10^{-14}
	1.0	0.4×10^{-14}	0.8×10^{-14}	0.5×10^{-14}

Sparse identification of fully-developed CIT

$$\underbrace{\frac{\varphi}{\tau_p} \langle u_f''' \rangle_p \langle u_p \rangle_p}_{\text{DP}} = \frac{\varphi}{\tau_p} C_g \langle u_p \rangle_p^2$$

Exact expression

Existing Closure

Closure Discovered by Sparse Regression

$$= -8.9 \frac{\varepsilon_f}{k_f} \langle u_f'''^2 \rangle + \frac{\varphi}{\tau_p} \left((0.6 - 1.9 \langle \alpha_p \rangle - 0.02 Re_p) \langle u_f'''^2 \rangle + 0.1 \langle v_f'''^2 \rangle \right) + \frac{\varphi}{\tau_p} \left(0.7 \langle u_p''^2 \rangle_p + 0.6 \langle v_p''^2 \rangle_p \right) - 0.4 Re_p \frac{\varphi}{\tau_p} \langle u_f'''^2 \rangle^{\frac{1}{2}} \langle u_p''^2 \rangle_p^{\frac{1}{2}}$$

Existing Model Error

		0.001	$\langle \alpha_p \rangle$ 0.01	0.1
Re_p	0.1	0.98	0.98	0.96
	0.3	0.97	0.98	0.95
	1.0	1.07	0.86	0.96

Sparse Identification Error

		0.001	$\langle \alpha_p \rangle$ 0.01	0.1
Re_p	0.1	1.5×10^{-14}	0.2×10^{-14}	0.04×10^{-14}
	0.3	2.2×10^{-14}	0.1×10^{-14}	0.01×10^{-14}
	1.0	0.3×10^{-14}	0.2×10^{-14}	0.02×10^{-14}

Homogeneous shear turbulence

The exact Reynolds-Stress equations for shear turbulence⁷ are

$$\frac{d}{dt}\langle u_i u_j \rangle = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\varepsilon\delta_{ij}$$

where

- \mathcal{P}_{ij} – production of TKE
- \mathcal{R}_{ij} – pressure rate-of-strain (redistribution of TKE)
- ε – dissipation of TKE

and unclosed terms are \mathcal{R}_{ij} and ε .

[7] Pope, S. (2012)

Homogeneous shear turbulence

A widely accepted closure for the pressures rate-of-strain term, is the LLR-IP model⁷

$$\mathcal{R}_{ij} = -C_R \frac{\varepsilon}{k} \left(\langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij} \right) - C_2 \left(\mathcal{P}_{ij} - \frac{2}{3} \mathcal{P} \delta_{ij} \right)$$

with typical values $C_R = 1.8$ and $C_2 = 3/5$.

[7] Pope, S. (2012)